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Three- and Five-Dimensional Considerations of the Geometry of Einstein's *g-Unified Field Theory

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We study the algebraic and differential geometric structures of three- and five-dimensional *g-unified field theory, with emphasis on the five-dimensional *g-unified field theory, in which we derive a new set of powerful recurrence relations which hold in a five-dimensional generalized Riemannian manifold X_5, prove a necessary and sufficient condition for the existence and uniqueness of the solution of the system of the Einstein equations in the first two classes, and find a precise tensorial representation of the Einstein connection $\Gamma^{\nu}_{\lambda\mu}$ in terms of $*g^{\lambda\nu}$.

1. INTRODUCTION

In Appendix II to his last book Einstein (1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory was physical, its exposition was mainly geometrical. It may be characterized as a set of geometrical postulates for the space-time X_4. The geometrical consequences of these postulates were not developed very far by Einstein.

Characterizing Einstein's unified field theory as a set of geometrical postulates for X_4, Hlavatý (1957) gave its mathematical foundation for the first time. The geometrical consequences of these postulates have been developed quite far, mainly by Hlavatý.

Generalizing X_4 to n-dimensional generalized Riemannian space X_n, Wrede (1958) studied Principles A and B given below. But his solution of our (2.8b) is not surveyable, probably due to the complexity of the higher dimensions. Later, Mishra (1959) also investigated the n-dimensional generalization of Principle A, using n-dimensional recurrence relations. The lower dimensional cases of the *usual* Einstein unified field theory were

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investigated by many authors. The four-dimensional case was studied by Hlavatý (1957) and many other geometricians, the two-dimensional case by Jakubowicz (1969) and Chung *et al.* (1983), and the three-dimensional case by Chung *et al.* (1979, 1980, 1981a,b).

The four-dimensional $^*g$-unified field theory in X_4 was first introduced by Chung (1963). This theory is convenient for investigating Principle C given below. Later, he generalized this theory to the n-dimensional case (Chung, 1982).

The purpose of the present paper is to study the structure of the three- and five-dimensional $^*g$-unified field theory. All our investigations and considerations are concerned with Principles A and B.

This paper contains six sections. Section 2 introduces some preliminary notations, concepts, and results in X_n. Section 3 deals with some relations in X_n, which will be needed in subsequent considerations. Section 4 is devoted exclusively to the algebraic and differential geometric structures of three-dimensional $^*g$-unified field theory for all classes in X_3. In Section 5 we investigate and study similar topics as in the previous section for the first two classes of five-dimensional $^*g$-unified field theory in X_5, mainly considering the derivation of a set of special recurrence relations and the solutions of the system of Einstein equations. In the last section we investigate mainly the solution of the Einstein equations for the third class of n-dimensional $^*g$-unified field theory when $n \geq 4$.

All considerations in the present paper are for all possible classes and indices of inertia.

2. PRELIMINARIES

This section is a brief collection of basic concepts, notations, and results needed in further considerations. All considerations in this section are for general $n > 1$. For these results see Chung (1982), Hlavatý, (1957), Mishra (1959), and Wrede (1958).

2.1. n-Dimensional $^*g$-Unified Field Theory

The n-dimensional $^*g$-unified field theory (n-$^*g$-UFT hereafter), originally suggested by Hlavatý (1957) and systematically introduced by Chung (1963), is based on the following three principles.

Principle A. Let X_n be an n-dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformation $x^\nu \rightarrow x^{\nu'}$, for which

$$\left|\frac{\partial x'}{\partial x}\right| \neq 0 \tag{2.1}$$

In the *usual* Einstein n-dimensional unified field theory (n-g-UFT hereafter), the manifold X_n is endowed with a general real, nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into a symmetric part $h_{\lambda\mu}$ and a skew-symmetric part $k_{\lambda\mu}$,[3]

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}, \qquad |g_{\lambda\mu}| \neq 0, \qquad |h_{\lambda\mu}| \neq 0 \tag{2.2}$$

However, the algebraic structure in our n-$^*g$-UFT is imposed on X_n by the basic real tensor $^*g^{\lambda\nu}$ defined by

$$g_{\lambda\mu}\,{}^*g^{\lambda\nu} = g_{\mu\lambda}\,{}^*g^{\nu\lambda} = \delta^\nu_\mu \tag{2.3}$$

It may also be split into a symmetric part $^*h^{\lambda\nu}$ and a skew-symmetric part $^*k^{\lambda\nu}$:

$$^*g^{\lambda\nu} = {}^*h^{\lambda\nu} + {}^*k^{\lambda\nu} \tag{2.4a}$$

$$|{}^*g^{\lambda\nu}| \neq 0, \qquad |{}^*h^{\lambda\nu}| \neq 0 \tag{2.4b}$$

In virtue of the second relation of (2.4b), we may define a unique tensor $^*h_{\lambda\mu}$ by

$$^*h_{\lambda\mu}\,{}^*h^{\lambda\nu} = \delta^\nu_\mu \tag{2.5}$$

In our n-$^*g$-UFT, we use both $^*h_{\lambda\mu}$ and $^*h^{\lambda\nu}$ as the tensors for raising and/or lowering indices of all tensors defined in X_n in the usual manner, with the exception of the tensors $g_{\lambda\mu}$, $h_{\lambda\mu}$, and $k_{\lambda\mu}$ in order to avoid notational confusion. We then have

$$^*k_{\lambda\mu} = {}^*k^{\alpha\beta}\,{}^*h_{\lambda\alpha}\,{}^*h_{\mu\beta}, \qquad {}^*g_{\lambda\mu} = {}^*g^{\alpha\beta}\,{}^*h_{\lambda\alpha}\,{}^*h_{\mu\beta} \tag{2.6a}$$

so that

$$^*g_{\lambda\mu} = {}^*h_{\lambda\mu} + {}^*k_{\lambda\mu} \tag{2.6b}$$

Principle B. The differential geometric structure is imposed on X_n by the tensor $^*g^{\lambda\nu}$ by means of a connection $\Gamma^\nu_{\lambda\mu}$ that obeys the transformation rule

$$\frac{\partial^2 x^\nu}{\partial x^{\lambda'}\,\partial x^{\mu'}} = \Gamma^{\nu'}_{\lambda'\mu'}\frac{\partial x^\nu}{\partial x^{\nu'}} - \Gamma^\nu_{\lambda\mu}\frac{\partial x^\lambda}{\partial x^{\lambda'}}\frac{\partial x^\mu}{\partial x^{\mu'}} \tag{2.7}$$

and satisfies the system of equations

$$D_\omega\,{}^*g^{\lambda\nu} = -2S^{\nu}_{\omega\alpha}\,{}^*g^{\lambda\alpha} \tag{2.8a}$$

[3]Throughout the paper, Greek indices are used for holonomic components of a tensor, Roman indices for the nonholonomic components of a tensor. In X_n all indices take the values $1, \ldots, n$ and follow summation convention with the exception of nonholonomic indices x, y, z, t.

where D_ω denotes the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and

$$S^\nu_{\lambda\mu} = \Gamma^\nu_{[\lambda\mu]} \tag{2.9}$$

is the torsion tensor of $\Gamma^\nu_{\lambda\mu}$. In virtue of (2.3), the system (2.8a) is equivalent to the original system of Einstein equations

$$D_\omega g_{\lambda\mu} = 2S^\alpha_{\omega\mu} g_{\lambda\alpha} \tag{2.8b}$$

Principle C. In order to obtain $^*g^{\lambda\nu}$ involved in the solution for $\Gamma^\nu_{\lambda\mu}$, certain conditions are imposed. These conditions may be condensed to

$$S_\lambda = S^\alpha_{\lambda\alpha} = 0, \qquad R_{\mu\lambda} = R^\alpha_{\alpha\mu\lambda} = \partial_{[\mu} V_{\lambda]}$$

where V_λ is an arbitrary vector, and

$$R^\nu_{\omega\mu\lambda} = 2\partial_{[\mu}\Gamma^\nu_{|\lambda|\omega]} + 2\Gamma^\alpha_{\lambda[\omega}\Gamma^\nu_{|\alpha|\mu]}$$

The following remarks should be added concerning the usual n-g-UFT.

Remark 2.1. In n-g-UFT, the algebraic structure is imposed on X_n by the tensor $g_{\lambda\mu}$, and the tensor $h_{\lambda\mu}$ and its inverse $h^{\lambda\nu}$ are used for raising and/or lowering indices in X_n. In this theory the differential geometric structure is imposed on X_n by $g_{\lambda\mu}$ by means of the same connection $\Gamma^\nu_{\lambda\mu}$ satisfying (2.8b).

Remark 2.2. When the system (2.8) admits a unique solution, the connection $\Gamma^\nu_{\lambda\mu}$ will be represented in terms of the tensor

$$\begin{array}{ll} ^*g^{\lambda\nu} & \text{in } n\text{-}^*g\text{-UFT} \\ g_{\lambda\mu} & \text{in } n\text{-}g\text{-UFT} \end{array}$$

In our further considerations, the following scalars, tensors, abbreviations, and notations are frequently used:

$$\mathfrak{g} = |^*g_{\lambda\mu}| \neq 0, \qquad \mathfrak{h} = |^*h_{\lambda\mu}| \neq 0, \qquad \mathfrak{k} = |^*k_{\lambda\mu}| \tag{2.10a}$$

$$g = \mathfrak{g}/\mathfrak{h}, \qquad k = \mathfrak{k}/\mathfrak{h} \tag{2.10b}$$

$$K_p = {}^*k^{\alpha_1}_{[\alpha_1} \cdots {}^*k^{\alpha_p}_{\alpha_p]} \qquad (p = 0, 1, 2, \cdots) \tag{2.10c}$$

$$^{(0)*}k^\nu_\lambda = \delta^\nu_\lambda, \qquad ^{(1)*}k^\nu_\lambda = {}^*k^\nu_\lambda, \qquad ^{(p)*}k^\nu_\lambda = {}^{(p-1)*}k^\alpha_\lambda\, {}^*k^\nu_\alpha \tag{2.10d}$$

$$K_{\omega\mu\nu} = \nabla_\nu {}^*k_{\omega\mu} + \nabla_\omega {}^*k_{\nu\mu} + \nabla_\mu {}^*k_{\omega\nu} \tag{2.10e}$$

$$\sigma = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \tag{2.10f}$$

Here ∇_ω is the symbolic vector of the covariant derivative with respect to the Christoffel symbols $^*\{{}_{\lambda\mu}^{\nu}\}$ defined by $^*h_{\lambda\mu}$ in the usual way. The scalars and tensors introduced in (2.10) satisfy

$$K_0 = 1; \qquad K_n = k \text{ if } n \text{ is even}; \qquad K_p = 0 \text{ if } p \text{ is odd} \tag{2.11a}$$

$$g = 1 + K_2 + \cdots + K_{n-\sigma} \tag{2.11b}$$

$$^{(p)*}k_{\lambda\mu} = (-1)^p\, {}^{(p)*}k_{\mu\lambda}, \qquad {}^{(p)*}k^{\lambda\nu} = (-1)^p\, {}^{(p)*}k^{\nu\lambda} \tag{2.11c}$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\lambda}$ skew-symmetric in the first two indices by T:

$$\overset{pqr}{T} = \overset{pqr}{T}_{\omega\mu\lambda} = T_{\alpha\beta\gamma}\, {}^{(p)*}k_\omega^\alpha\, {}^{(q)*}k_\mu^\beta\, {}^{(r)*}k_\lambda^\gamma \tag{2.12a}$$

$$T = T_{\omega\mu\lambda} = \overset{000}{T} \tag{2.12b}$$

$$2\overset{pqr}{T}_{\omega[\lambda\mu]} = \overset{pqr}{T}_{\omega\lambda\mu} - \overset{pqr}{T}_{\omega\mu\lambda}, \qquad 2\overset{(pq)r}{T}_{\omega\lambda\mu} = \overset{pqr}{T}_{\omega\lambda\mu} + \overset{qpr}{T}_{\omega\lambda\mu}, \qquad \text{etc.}$$

We then have

$$\overset{pqr}{T}_{\omega\lambda\mu} = -\overset{qpr}{T}_{\lambda\omega\mu} \tag{2.13}$$

2.2. Algebraic Preliminaries

In this section several algebraic concepts and results in n-$^*g$-UFT are introduced.

Definition 2.3. The tensor $^*g_{\lambda\mu}$ (or $^*k_{\lambda\mu}$) is said to be:

1. Of the first class if $K_{n-\sigma} \neq 0$
2. Of the second class with jth category ($j \geq 1$) if

$$K_{2j} \neq 0, \qquad K_{2j+2} = K_{2j+4} = \cdots = K_{n-\sigma} = 0$$

3. Of the third class if $K_2 = K_4 = \cdots = K_{n-\sigma} = 0$.

The solution of the system of equations (2.8a) is most conveniently brought about in a nonholonomic frame of reference, which may be introduced by the projectivity

$$MA^\nu = {}^*k_\mu^\nu A^\mu \qquad (M \text{ a scalar}) \tag{2.14}$$

Definition 2.4. An eigenvector A^ν of $^*k_{\lambda\mu}$ that satisfies (2.14) is called a basic vector in X_n, and the corresponding eigenvalue M is termed a basic scalar.

The basic scalars M are solutions of the characteristic equation

$$M^{\sigma}(M^{n-\sigma}+K_2M^{n-2-\sigma}+\cdots+K_{n-2-\sigma}M^2+K_{n-\sigma})=0 \tag{2.15}$$

In each class the nonholonomic frame of reference may be constructed as follows.

Case 1. First and second classes. In the first two classes we have a set of n linearly independent basic vectors $\underset{i}{A}^{\nu}$ $(i=1,\ldots,n)$ and a unique reciprocal set $\overset{i}{A}_{\lambda}$ $(i=1,\ldots,n)$ satisfying

$$\overset{j}{A}_{\lambda}\underset{i}{A}^{\lambda}=\delta^j_i, \qquad \overset{i}{A}_{\lambda}\underset{i}{A}^{\nu}=\delta^{\nu}_{\lambda} \tag{2.16}$$

With these two sets of vectors we can construct a nonholonomic frame of reference as follows:

Definition 2.5. If $T^{\nu\ldots}_{\lambda\ldots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

$$T^{i\ldots}_{j\ldots}=T^{\nu\ldots}_{\lambda\ldots}\overset{i}{A}_{\nu}\ldots\underset{j}{A}^{\lambda}\ldots \tag{2.17a}$$

An easy inspection shows that

$$T^{\nu\ldots}_{\lambda\ldots}=T^{i\ldots}_{j\ldots}\underset{i}{A}^{\nu}\ldots\overset{j}{A}_{\lambda}\ldots \tag{2.17b}$$

Furthermore, if $\underset{x}{M}$ is the basic scalar corresponding to $\underset{x}{A}^{\nu}$, then the nonholonomic components of ${}^{(p)*}k_{\lambda}^{\nu}$ are given by

$${}^{(p)*}k_x^i=\underset{x}{M}^p\delta^i_x, \qquad {}^{(p)*}k_{xi}=\underset{x}{M}^p\,{}^*h_{xi}, \qquad {}^{(p)*}k^{xi}=\underset{x}{M}^p\,{}^*h^{xi} \tag{2.18a}$$

Without loss of generality we may choose the nonholonomic components of ${}^*h_{\lambda\mu}$ as

$$\begin{aligned} &{}^*h_{12}={}^*h_{34}=\cdots={}^*h_{n-1-\sigma,n-\sigma}=1\\ &\sigma{}^*h_{ni_0}=\delta^1_{\sigma}, \qquad \text{remaining } {}^*h_{ij}=0 \end{aligned} \tag{2.18b}$$

where the index i_0 is taken so that $|{}^*h_{ij}|\neq 0$ when n is odd.

Case 2. Third class. In the third class, the above frame is not available, since all $\underset{x}{M}=0$ in this case, and hence another nonholonomic frame of reference should be constructed. In the frame of reference of the third class, the basic scalar satisfy

$$\underset{1}{A}^{\lambda}\,{}^*k_{\lambda}^{\nu}=\underset{2}{A}^{\nu}, \qquad \underset{2}{A}^{\lambda}\,{}^*k_{\lambda}^{\nu}=\underset{4}{A}^{\nu}, \qquad \underset{f}{A}^{\lambda}\,{}^*k_{\lambda}^{\nu}=0, \quad f=3,4,\ldots,n \tag{2.19a}$$

for $n \geq 4$. Without loss of generality we may choose the nonholonomic components of $^*h_{\lambda\mu}$, when $n \geq 4$, as

$$\begin{gathered} {}^*h_{22} = {}^*h_{33} = {}^*h_{55} = {}^*h_{66} = \cdots = {}^*h_{nn} = -{}^*h_{14} = -1 \\ \text{remaining } {}^*h_{ij} = 0 \end{gathered} \tag{2.19b}$$

We then have

$$\begin{gathered} {}^*k_{21} = {}^*k_1^2 = {}^*k_2^4 = {}^*k^{42} = 1 \\ \text{remaining } {}^*k_{ij}, {}^*k_i^j, {}^*k^{ij} = 0 \\ {}^{(p)*}k_1^4 = {}^{(p)*}k_{11} = {}^{(p)*}k^{44} = \delta_2^p \\ \text{remaining } {}^{(p)*}k_{ij}, {}^{(p)*}k_i^j, {}^{(p)*}k^{ij} = 0 \qquad (p \geq 2) \end{gathered} \tag{2.19c}$$

2.3. Differential Geometric Preliminaries

In this section we exhibit several results needed in our subsequent considerations for the solution of (2.8a).

If the system (2.8a) admits $\Gamma^\nu_{\lambda\mu}$, it must be of the form

$$\Gamma^\nu_{\lambda\mu} = {}^*\{{}^{\,\nu}_{\lambda\mu}\} + S^\nu_{\lambda\mu} + U^\nu_{\lambda\mu} \tag{2.20}$$

where

$$U_{\nu\lambda\mu} = \overset{100}{S}_{(\lambda\mu)\nu} + 2\overset{(10)0}{S}_{\nu(\lambda\mu)} \tag{2.21}$$

The above two relations show that our problem of determining $\Gamma^\nu_{\lambda\mu}$ in terms of $^*g^{\lambda\nu}$ is reduced to that of studying the tensor $S^\nu_{\lambda\mu}$.

On the other hand, it has also been shown that the tensor $S^\nu_{\lambda\mu}$ satisfies

$$S = B - 3\overset{(110)}{S} \tag{2.22}$$

where

$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}\, {}^*k^\alpha_{\omega]}\, {}^*k^\beta_\nu \tag{2.23}$$

In subsequent sections we start with relation (2.22) to solve the system (2.8a).

Furthermore, for the first two classes, the nonholonomic solution of (2.22) is given by

$$\underset{xyz}{M} S_{xyz} = B_{xyz} \tag{2.24a}$$

or equivalently

$$4\underset{xyz}{M} S_{xyz} = (2 + \underset{z}{M}\underset{x}{M} + \underset{z}{M}\underset{y}{M}) K_{xyz} + \underset{z}{M}(\underset{x}{M} + \underset{z}{M}) K_{zxy} + \underset{z}{M}(\underset{y}{M} + \underset{z}{M}) K_{yzx} \tag{2.24b}$$

where

$$\underset{xyz}{M} = 1 + \underset{x}{M}\underset{y}{M} + \underset{y}{M}\underset{z}{M} + \underset{z}{M}\underset{x}{M} \tag{2.25}$$

Therefore, in virtue of (2.24), we see that a necessary and sufficient conditions for the system (2.8a) to have a unique solution in the first two classes is

$$\underset{xyz}{M} \neq 0 \qquad \text{for all} \quad x, y, z \tag{2.26}$$

3. SOME RELATIONS IN n-*g-UFT

In this section we derive several useful relations in X_n, mainly recurrence relations. All considerations in this section are for general $n > 1$.

Theorem 3.1. (All classes.) We have

$$\underset{i}{A}{}^{\nu} = \overset{j}{A}_{\lambda} {}^*h_{ij} {}^*h^{\lambda\nu}, \qquad \overset{j}{A}_{\lambda} = \underset{i}{A}{}^{\nu} {}^*h^{ij} {}^*h_{\lambda\nu} \tag{3.1}$$

Proof. In virtue of (2.5), (2.16), and (2.17a), the first relation of (3.1) is given as follows:

$$\overset{j}{A}_{\lambda} {}^*h_{ij} {}^*h^{\lambda\nu} = \overset{j}{A}_{\lambda} ({}^*h_{\alpha\beta} \underset{i}{A}{}^{\alpha} \underset{j}{A}{}^{\beta}) {}^*h^{\lambda\nu} = \underset{i}{A}{}^{\alpha} ({}^*h_{\alpha\beta} {}^*h^{\lambda\nu}) \delta^{\beta}_{\lambda} = \underset{i}{A}{}^{\nu} \quad \blacksquare$$

Theorem 3.2. (First and second classes.) In the first two classes, the tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, satisfies

$$\overset{(pq)r}{T}_{\omega\mu\nu} = \sum_{x,y,z} T_{xyz} \underset{x}{M}{}^{(p} \underset{y}{M}{}^{q)} \underset{z}{M}{}^{r} \overset{x}{A}_{\omega} \overset{y}{A}_{\mu} \overset{z}{A}_{\nu} \tag{3.2a}$$

$$\overset{r(pq)}{T}_{\nu[\omega\mu]} = \sum_{x,y,z} T_{x[yz]} \underset{y}{M}{}^{(p} \underset{z}{M}{}^{q)} \underset{x}{M}{}^{r} \overset{x}{A}_{\nu} \overset{y}{A}_{\omega} \overset{z}{A}_{\mu} \tag{3.2b}$$

where

$$2\underset{x}{M}{}^{(p} \underset{y}{M}{}^{q)} = \underset{x}{M}{}^{p} \underset{y}{M}{}^{q} + \underset{x}{M}{}^{q} \underset{y}{M}{}^{p} \tag{3.2c}$$

Proof. In virtue of (2.17b) and (2.18a), our assertion (3.2a) takes the form

$$\begin{aligned}
\overset{(pq)r}{T}_{\omega\mu\nu} &= \sum_{x,y,z} \overset{(pq)r}{T}_{xyz} \overset{x}{A}_{\omega} \overset{y}{A}_{\mu} \overset{z}{A}_{\nu} \\
&= \sum_{x,y,z} \tfrac{1}{2} T_{ijk} ({}^{(p)*}k^{i}_{x} \, {}^{(q)*}k^{j}_{y} + {}^{(q)*}k^{i}_{x} \, {}^{(p)*}k^{j}_{y}) \, {}^{(r)*}k^{k}_{z} \overset{x}{A}_{\omega} \overset{y}{A}_{\mu} \overset{z}{A}_{\nu} \\
&= \tfrac{1}{2} \sum_{x,y,z} T_{xyz} (\underset{x}{M}{}^{p} \underset{y}{M}{}^{q} + \underset{x}{M}{}^{q} \underset{y}{M}{}^{p}) \underset{z}{M}{}^{r} \overset{x}{A}_{\omega} \overset{y}{A}_{\mu} \overset{z}{A}_{\nu}
\end{aligned}$$

The second relation can be proved in similar ways. $\blacksquare$

Theorem 3.3. (First class.) The main recurrence relation in the first class is

$$ {}^{(n+p)*}k_\lambda^\nu + K_2\, {}^{(n+p-2)*}k_\lambda^\nu + \cdots + K_{n-\sigma-2}\, {}^{(p+\sigma+2)*}k_\lambda^\nu + K_{n-\sigma}\, {}^{(p+\sigma)*}k_\lambda^\nu = 0 \tag{3.3a} $$

which may be condensed to

$$ \sum_{f=0}^{n-\sigma} K_f\, {}^{(n+p-f)*}k_\lambda^\nu = 0 \qquad (p = 0, 1, 2, \ldots) \tag{3.3b} $$

Proof. Let $\underset{x}{M}$ be a basic scalar. Then, in virtue of (2.15), we have

$$ \sum_{f=0}^{n-\sigma} K_f \underset{x}{M}^{n-f} = 0 \tag{3.4} $$

Multiplying by δ_x^i on both sides of (3.4) and making use of (2.18a), we have

$$ \sum_{f=0}^{n-\sigma} K_f\, {}^{(n-f)*}k_x^i = 0 \tag{3.5a} $$

whose holonomic form is

$$ \sum_{f=0}^{n-\sigma} K_f\, {}^{(n-f)*}k_\lambda^\alpha = 0 \tag{3.5b} $$

The relation (3.3) immediately follows by multiplying by ${}^{(p)*}k_\alpha^\nu$ on both sides of (3.5b). ■

Theorem 3.4. (Second class with jth category.) The main recurrence relation in the second class with jth category is

$$ {}^{(2j+p)*}k_\lambda^\nu + K_2\, {}^{(2j+p-2)*}k_\lambda^\nu + \cdots + K_{2j}\, {}^{(p)*}k_\lambda^\nu = 0 \tag{3.6a} $$

which may be condensed to

$$ \sum_{f=0}^{2j} K_f\, {}^{(2j+p-f)*}k_\lambda^\nu = 0 \qquad (p = 1, 2, \ldots) \tag{3.6b} $$

Proof. When $^*g_{\lambda\mu}$ belongs to the second class with jth category, the characteristic equation (2.15) is reduced to

$$ \sum_{f=0}^{2j} K_f M^{n-f} = M^{n-2j} \sum_{f=0}^{2j} K_f M^{2j-f} = 0 \tag{3.7a} $$

in virtue of Definition 2.3. Hence, if $\underset{x}{M}$ is a root of (3.7a), it satisfies

$$ 0 = \underset{x}{M} \sum_{f=0}^{2j} K_f \underset{x}{M}^{2j-f} = \sum_{f=0}^{2j} K_f \underset{x}{M}^{2j-f+1} \tag{3.7b} $$

Multiplying by δ_x^i on both sides of (3.7b) and making use of (2.18a) we have

$$\sum_{f=0}^{2j} K_f \, {}^{(2j-f+1)*}k_x^i = 0 \tag{3.8a}$$

with holonomic form

$$\sum_{f=0}^{2j} K_f \, {}^{(2j-f+1)*}k_\lambda^\alpha = 0 \tag{3.8b}$$

The relation (3.6) follows by multiplying by ${}^{(p-1)*}k_\alpha^\nu$ on both sides of (3.8b). ■

Remark 3.5. As a particular case of (3.6), we have

$$ {}^{(p+2)*}k_\lambda^\nu + K_2 \, {}^{(p)*}k_\lambda^\nu = 0 \qquad (p = 1, 2, \ldots) \tag{3.9}$$

when ${}^*g_{\lambda\mu}$ belongs to the second class with the first category.

Theorem 3.6. (Third class.) The main recurrence relation in the third class is

$$ {}^{(p)*}k_\lambda^\nu = 0 \qquad (p = 3, 4, \ldots) \tag{3.10}$$

Proof. Relation (3.10) is a direct consequence of putting $K_2 = 0$ in (3.9). ■

In the following theorem, we prove two relations in X_n that hold for all classes. These relations are used in subsequent sections, when we are concerned with the solution of (2.8a).

Theorem 3.7. (All classes.) We have

$$\overset{(pq)r}{B} = \overset{(pq)r}{S} + \overset{(p'q')r}{S} + \overset{(p'q)r'}{S} + \overset{(pq')r'}{S} \tag{3.11}$$

$$2\overset{(pq)r}{B}_{\omega\mu\nu} = \overset{(pq)r}{K}_{\omega\mu\nu} + \overset{r''(pq)}{K}_{\nu[\omega\mu]} + \tfrac{1}{2}\big(\overset{(pq')r'}{K}_{\omega\mu\nu} + \overset{(p'q)r'}{K}_{\omega\mu\nu} + \overset{r'p'q}{K}_{\nu[\omega\mu]} + \overset{r'q'p}{K}_{\nu[\omega\mu]}\big) \tag{3.12}$$

where

$$p' = p + 1, \qquad q' = q + 1, \qquad r' = r + 1, \qquad r'' = r + 2 \tag{3.13}$$

Proof. In virtue of (2.22) and (2.12), the first relation (3.11) may be obtained in the following way:

$$\begin{aligned}
\overset{(pq)r}{B} = \overset{(pq)r}{B}_{\omega\mu\nu} &= \tfrac{1}{2}B_{\alpha\beta\gamma}\big({}^{(p)*}k_\omega^\alpha \, {}^{(q)*}k_\mu^\beta + {}^{(q)*}k_\omega^\alpha \, {}^{(p)*}k_\mu^\beta\big) \, {}^{(r)*}k_\nu^\gamma \\
&= \tfrac{1}{2}\big(S_{\alpha\beta\gamma} + S_{\varepsilon\eta\gamma} \, {}^*k_\alpha^\varepsilon \, {}^*k_\beta^\eta + S_{\varepsilon\beta\eta} \, {}^*k_\alpha^\varepsilon \, {}^*k_\gamma^\eta + S_{\alpha\varepsilon\eta} \, {}^*k_\beta^\varepsilon \, {}^*k_\gamma^\eta\big) \\
&\quad \times \big({}^{(p)*}k_\omega^\alpha \, {}^{(q)*}k_\mu^\beta + {}^{(q)*}k_\omega^\alpha \, {}^{(p)*}k_\mu^\beta\big) \, {}^{(r)*}k_\nu^\gamma
\end{aligned}$$

After a lengthy calculation, we note that the right-hand side of the above equation is equal to that of (3.11). Similarly, we can verify (3.12) by using (2.12) and (2.23). ■

4. 3-*g-UFT

In 3-*g-UFT we have only two classes; the first class, when $K_2 \neq 0$, and the third class, when $K_2 = 0$. In 3-*g-UFT relation (2.11b) gives

$$g = 1 + K_2 \tag{4.1}$$

This section is concerned with the three-dimensional *g-unified field theory, with emphasis on the nonholonomic frame of reference, recurrence relations, and the Einstein connection in the first class. Hence, in this section we restrict ourselves to $n = 3$.

4.1. Basic Vectors and Nonholonomic Frame of Reference in the First Class

Theorem 4.1. The basic scalars are

$$\underset{1}{M} = -\underset{2}{M} = (-K_2)^{1/2}, \qquad \underset{3}{M} = 0 \tag{4.2}$$

Proof. In 3-*g-UFT, the characteristic equation (2.15) is reduced to

$$M(M^2 + K_2) = 0 \tag{4.3}$$

from which our assertion follows. ■

Theorem 4.2. There are three linearly independent basic vectors $\underset{1}{A}^\nu, \underset{2}{A}^\nu, \underset{3}{A}^\nu$, and they have the following properties:

(a) They are defined up to an arbitrary factor of proportionality.
(b) $\underset{1}{A}^\nu$ and $\underset{2}{A}^\nu$ are null vectors and $\underset{3}{A}^\nu$ is nonnull.
(c) $\underset{3}{A}^\nu$ is perpendicular to $\underset{1}{A}^\nu$ and $\underset{2}{A}^\nu$.
(d) $\underset{1}{A}^\nu$ and $\underset{2}{A}^\nu$ satisfy the condition

$$h_{\lambda\mu} \underset{1}{A}^\lambda \underset{2}{A}^\mu \neq 0 \tag{4.4}$$

Proof. Since the basic scalars $\underset{i}{M}$ are all distinct, (2.14) admits three linearly independent basic vectors $\underset{i}{A}^\nu$, which are defined up to an arbitrary factor of proportionality. The first half of statement (b) is a consequence of (2.14), (4.2), and

$$\underset{x}{M}{}^*h_{\lambda\mu}\underset{x}{A}^\lambda A^\mu = {}^*k_{\lambda\mu}\underset{x}{A}^\lambda \underset{x}{A}^\mu = 0 \qquad (x=1,2)$$

Since $\underset{3}{M}+\underset{x}{M}\neq 0$ $(x=1,2)$, statement (c) follows from (2.14) in the following way:

$$\underset{x}{M}{}^*h_{\lambda\mu}\underset{x}{A}^\lambda \underset{3}{A}^\mu = {}^*k_{\lambda\mu}\underset{x}{A}^\lambda \underset{3}{A}^\mu = -\underset{3}{M}{}^*h_{\mu\lambda}\underset{x}{A}^\lambda \underset{3}{A}^\mu$$

In order to prove statement (d), consider a conic C with equation ${}^*h_{\lambda\mu}x^\lambda x^\mu = 0$ on a projective plane P_2. In virtue of statement (b), $\underset{1}{A}^\nu$ and $\underset{2}{A}^\nu$ are two different points on C, and ${}^*h_{\lambda\mu}\underset{1}{A}^\lambda = \underset{1}{A}_\mu$ is the tangent line to C at $\underset{1}{A}^\nu$. Since $|{}^*h_{\lambda\mu}|\neq 0$, C is nondegenerate. Consequently ${}^*h_{\lambda\mu}\underset{1}{A}^\lambda = \underset{1}{A}_\mu$ and $\underset{2}{A}^\mu$ are not incident; that is, ${}^*h_{\lambda\mu}\underset{1}{A}^\lambda \underset{2}{A}\neq 0$. In order to show the last half of statement (b), assume that ${}^*h_{\lambda\mu}\underset{3}{A}^\lambda \underset{3}{A}^\mu = 0$. Then, statement (c) gives

$$|{}^*h_{ij}| = 0 \Rightarrow |\underset{i}{A}^\nu|^2 = 0 \Rightarrow |\underset{i}{A}^\nu| = 0$$

which contradicts the linear independence of $\underset{i}{A}^\nu$. ■

Agreement 4.3. We may choose the factor of proportionality mentioned in Theorem 4.2 as

$${}^*h_{12} = {}^*h_{33} = 1 \tag{4.5}$$

This agreement is coincident with (2.18b).

In virtue of Theorem 4.2 and the above agreement, we have the following result:

Theorem 4.4. The nonholonomic components ${}^*h_{ij}$ and ${}^*h^{ij}$ are given by the matrix equation

$$(({}^*h_{ij})) = (({}^*h^{ij})) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.6}$$

Relation (3.1) together with (4.6) give the following result:

Theorem 4.5. We have

$$\underset{1}{A}^{\nu}=\overset{2}{A}^{\nu},\quad \underset{2}{A}^{\nu}=\overset{1}{A}^{\nu},\quad \underset{3}{A}^{\nu}=\overset{3}{A}^{\nu},\quad \overset{1}{A}_{\lambda}=\underset{2}{A}_{\lambda},\quad \overset{2}{A}_{\lambda}=\underset{1}{A}_{\lambda},\quad \overset{3}{A}_{\lambda}=\underset{3}{A}_{\lambda} \tag{4.7}$$

As an application of the nonholonomic frame of reference constructed in the above discussion, we have the following:

Theorem 4.6. The representations of $*h_{\lambda\mu}$, ${}^{(p)*}k_{\lambda}^{\nu}$, ${}^{(p)*}k_{\lambda\mu}$, and ${}^{(p)*}k^{\lambda\nu}$ in terms of basic vectors are given by

$$*h_{\lambda\mu}=2\overset{1}{A}_{(\lambda}\overset{2}{A}_{\mu)}+\overset{3}{A}_{\lambda}\overset{3}{A}_{\mu} \tag{4.8a}$$

$${}^{(p)*}k_{\lambda}^{\nu}=\underset{1}{M}^{p}(\overset{1}{A}_{\lambda}\underset{1}{A}^{\nu}+(-1)^{p}\overset{2}{A}_{\lambda}\underset{2}{A}^{\nu}) \tag{4.8b}$$

$${}^{(p)*}k_{\lambda\mu}=\begin{cases}2\underset{1}{M}^{p}\overset{1}{A}_{(\lambda}\overset{2}{A}_{\mu)} & \text{if } p \text{ is even}\\ 2\underset{1}{M}^{p}\overset{1}{A}_{[\lambda}\overset{2}{A}_{\mu]} & \text{if } p \text{ is odd}\end{cases} \tag{48c}$$

$${}^{(p)*}k^{\lambda\mu}=\begin{cases}2\underset{1}{M}^{p}\underset{1}{A}^{(\lambda}\underset{2}{A}^{\nu)} & \text{if } p \text{ is even}\\ 2\underset{1}{M}^{p}\underset{1}{A}^{[\lambda}\underset{2}{A}^{\nu]} & \text{if } p \text{ is odd}\end{cases} \tag{4.8d}$$

Proof. The representations (4.8) follow from (2.17b) in virtue of (4.2), (4.6), and (4.7). ■

4.2. Recurrence Relations in the First Class

In this section we derive several recurrence relations in addition to (3.3), which now in the first class of 3-*g-UFT assumes the form

$${}^{(p+3)*}k_{\lambda}^{\nu}=-K_{2}\,{}^{(p+1)*}k_{\lambda}^{\nu}\qquad (p=0,1,2,\ldots) \tag{4.9}$$

Theorem 4.7. The basic scalars $\underset{x}{M}$ satisfy $(x\neq y)$

$$\underset{x}{M}\underset{y}{M}(\underset{x}{M}+\underset{y}{M})=0 \tag{4.10a}$$

$$\underset{x}{M}\underset{y}{M}(\underset{x}{M}\underset{y}{M}-K_{2})=0 \tag{4.10b}$$

Proof. These relations follow easily from (4.2). ■

Theorem 4.8. (Recurrence relations.) If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then the following recurrence relations hold in the first class of 3-*g-UFT:

$$\overset{(12)r}{T}=0, \qquad \overset{22r}{T}=K_2\overset{11r}{T} \tag{4.11a}$$

$$\overset{r(12)}{T}_{\nu[\omega\mu]}=0, \qquad \overset{r22}{T}_{\nu[\omega\mu]}=K_2\overset{r11}{T}_{\nu[\omega\mu]} \tag{4.11b}$$

Proof. The above relations are consequences of (3.2) and (4.10). For example, the second relation of (4.11a) can be proved as follows:

$$\begin{aligned}\overset{22r}{T}=\overset{22r}{T}_{\omega\mu\nu}&=\sum_{x,y,z} T_{xyz}\underset{x}{M^2}\underset{y}{M^2}\underset{z}{M^r}\overset{x}{A}_\omega\overset{y}{A}_\mu\overset{z}{A}_\nu\\ &=K_2\sum_{x,y,z} T_{xyz}\underset{x}{M}\underset{y}{M}\underset{z}{M^r}\overset{x}{A}_\omega\overset{y}{A}_\mu\overset{z}{A}_\nu\\ &=K_2\overset{11r}{T}_{\omega\mu\nu}=K_2\overset{11r}{T} \quad \blacksquare\end{aligned}$$

4.3. Einstein Connection $\Gamma^{\nu}_{\lambda\mu}$ in the First Class

In this section we derive two surveyable tensorial representations of $S^{\nu}_{\lambda\mu}$ and hence $\Gamma^{\nu}_{\lambda\mu}$ in terms of $*g^{\lambda\nu}$, employing the recurrence relations (4.11) and the relations (3.11) and (3.12a).

Theorem 4.9. A necessary and sufficient condition for the system (2.8a) to admit a unique solution $\Gamma^{\nu}_{\lambda\mu}$ is that

$$1-(K_2)^2\neq 0 \tag{4.12}$$

Proof. Since $\underset{xyz}{M}$ defined by (2.26) is symmetric in x, y, and z and satisfies

$$\underset{33x}{M}=1, \qquad \underset{311}{M}=\underset{322}{M}=1-K_2, \qquad \underset{123}{M}=\underset{112}{M}=\underset{122}{M}=1+K_2$$

we have the condition (4.12) in virtue of (2.27). ■

Theorem 4.10. The system of equations (2.22) is reduced to a system of the following five equations:

$$\begin{aligned}B&=S+2\overset{(10)1}{S}+\overset{110}{S}\\ \overset{(10)1}{B}&=\overset{(10)1}{S}+\overset{(20)2}{S}+\overset{112}{S}\\ \overset{110}{B}&=(1+K_2)\overset{110}{S}\\ \overset{(20)2}{B}&=(K_2)^2\overset{(10)1}{S}+\overset{(20)2}{S}-K_2\overset{112}{S}\\ \overset{112}{B}&=(1+K_2)\overset{112}{S}\end{aligned} \tag{4.13}$$

Proof. This assertion follows from (3.11), using (4.9) and (4.11a). ■

Theorem 4.11. The tensors $\overset{(pq)r}{B}_{\omega\mu\nu}$ are given as linear combinations of $\overset{(pq)r}{K}_{\omega\mu\nu}$ as follows:

$$2\overset{(10)1}{B}_{\omega\mu\nu} = \overset{(10)1}{K}_{\omega\mu\nu} + \tfrac{1}{2}(\overset{(11)2}{K}_{\omega\mu\nu} + \overset{(20)2}{K}_{\omega\mu\nu} + \overset{211}{K}_{\nu[\omega\mu]} - K_2 \overset{101}{K}_{\nu[\omega\mu]})$$

$$2\overset{110}{B}_{\omega\mu\nu} = \overset{110}{K}_{\omega\mu\nu} \tag{4.14}$$

$$2\overset{(20)2}{B}_{\omega\mu\nu} = \overset{(20)2}{K}_{\omega\mu\nu} + \tfrac{1}{2}[(K_2)^2 \overset{(10)1}{K}_{\omega\mu\nu} - K_2 \overset{112}{K}_{\nu[\omega\mu]} - K_2 \overset{202}{K}_{\nu[\omega\mu]}]$$

$$2\overset{112}{B}_{\omega\mu\nu} = \overset{112}{K}_{\omega\mu\nu}$$

Proof. These relations may be obtained from (3.12) in virtue of (4.9) and (4.11). ■

Theorem 4.12. If the condition (4.12) is satisfied, the unique solution of (2.22) is given by

$$[1-(K_2)^2](S-B) = -2\overset{(10)1}{B} + (K_2 - 1)\overset{110}{B} + 2\overset{(20)2}{B} + 2\overset{112}{B} \tag{4.15a}$$

or equivalently

$$[1-(K_2)^2](2S_{\omega\mu\nu} - K_{\omega\mu\nu} - \overset{110}{K}_{\nu[\omega\mu]} - \overset{200}{K}_{\nu[\omega\mu]}$$
$$= -\overset{(10)1}{K}_{\omega\mu\nu} + \overset{112}{K}_{\omega\mu\nu} + \overset{(20)2}{K}_{\omega\mu\nu} - \overset{211}{K}_{\nu[\omega\mu]}$$
$$+(K_2 - 1)\overset{110}{K}_{\omega\mu\nu} + K_2(\overset{101}{K}_{\nu[\omega\mu]} - \overset{112}{K}_{\nu[\omega\mu]} - \overset{202}{K}_{\nu[\omega\mu]}). \tag{4.15b}$$

Proof. Relation (4.15a) is the solution of (4.13), while (4.15b) may be obtained by substituting (4.14) into (4.15a) and making use of recurrence relations. ■

Theorem 4.13. The tensor $U^{\nu}_{\lambda\mu}$ is given by

$$[1-(K_2)^2](U_{\nu\lambda\mu} - \overset{[10]0}{B}_{\lambda\mu\nu} - 2\overset{(10)0}{B}_{\nu(\lambda\mu)}$$
$$= -K_2(\overset{[10]2}{B}_{\lambda\mu\nu} + 2\overset{(10)2}{B}_{\nu(\lambda\mu)}) + (K_2 - 1)\overset{[21]0}{B}_{\lambda\mu\nu} + \overset{[02]1}{B}_{\lambda\mu\nu} + \overset{[21]2}{B}_{\lambda\mu\nu}$$
$$-2\overset{(20)1}{B}_{\nu(\kappa\mu)} - 2\overset{111}{B}_{\nu(\lambda\mu)} \tag{4.16a}$$

or equivalently

$$[1-(K_2)_2](2U_{\nu\lambda\mu} + \overset{[01]0}{K}_{\lambda\mu\nu} - 2\overset{(10)0}{K}_{\nu(\lambda\mu)}$$
$$= (K_2 - 1)\overset{[21]0}{K}_{\lambda\mu\nu} + \overset{[02]1}{K}_{\lambda\mu\nu} + K_2 \overset{[01]2}{K}_{\lambda\mu\nu}$$
$$-2(K_2 \overset{(10)2}{K}_{\nu(\lambda\mu)} - \overset{(20)1}{K}_{\nu(\lambda\mu)} - \overset{111}{K}_{\nu(\lambda\mu)}) \tag{4.16b}$$

Proof. The representations (4.16) are direct consequences of substituting (4.15) into (2.21). ■

Now that we have obtained the tensors $S^{\nu}_{\lambda\mu}$ and $U^{\nu}_{\lambda\mu}$ in terms of $^*g^{\lambda\nu}$, it is possible to determine $\Gamma^{\nu}_{\lambda\mu}$ by only substituting for S and U into (2.20).

4.4. The Third Class

In this section we discuss the algebraic and differential geometric properties of the third class of 3-$^*g$-UFT.

The third class of 3-$^*g$-UFT was studied by Chung *et al.* (1980, 1981a,b). Since most results in 3-$^*g$-UFT are similar to Chung's results in the third class, we simply state them without proof.

Theorem 4.14. There is a one-parameter set consisting of three linearly independent real vectors $\underset{1}{A}^{\nu}$, $\underset{2}{A}^{\nu}$, $\underset{3}{A}^{\nu}$, which satisfy the following relations:

$$^*k^{\nu}_{\lambda}\underset{2}{A}^{\lambda}=0, \qquad ^*k^{\nu}_{\lambda}\underset{1}{A}^{\lambda}=\underset{2}{A}^{\nu}, \qquad ^*k^{\nu}_{\lambda}\underset{3}{A}^{\lambda}=\underset{1}{A}^{\nu} \tag{4.17}$$

$$((^*h_{ij}))=((^*h^{ij}))=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & 0\end{pmatrix} \tag{4.18}$$

Theorem 4.15. We have

$$\begin{aligned}&\underset{1}{A}^{\nu}=\overset{1}{A}^{\nu}, \qquad \underset{2}{A}^{\nu}=-\overset{3}{A}^{\nu}, \qquad \underset{3}{A}^{\nu}=-\overset{2}{A}^{\nu}\\ &\overset{1}{A}_{\lambda}=\underset{1}{A}_{\lambda}, \qquad \overset{2}{A}_{\lambda}=-\underset{3}{A}_{\lambda}, \qquad \overset{3}{A}_{\lambda}=-\underset{2}{A}_{\lambda}\end{aligned} \tag{4.19}$$

Theorem 4.16. The nonholonomic components $^{(p)*}k_i^j$, $^{(p)*}k_{ij}$, and $^{(p)*}k^{ij}$ are given by

$$\begin{aligned}&^*k_1^2={}^*k_3^1={}^*k_{31}=-{}^*k_{13}={}^*k^{12}=-{}^*k^{21}=1\\ &^{(p)*}k_3^2=-{}^{(p)*}k_{33}=-{}^{(p)*}k^{22}=\delta_2^p \qquad (p=1,2,\ldots)\\ &\text{remaining}=0\end{aligned} \tag{4.20}$$

Theorem 4.17. The tensors $^*h_{\lambda\mu}$, $^{(p)*}k^{\nu}_{\lambda}$, $^{(p)*}k_{\lambda\mu}$, and $^{(p)*}k^{\lambda\nu}$ may be expressed in terms of the basic vectors as follows:

$$^*h_{\lambda\mu}=\overset{1}{A}_{\lambda}\overset{1}{A}_{\mu}-2\overset{2}{A}_{(\lambda}\overset{3}{A}_{\mu)} \tag{4.21a}$$

$$ {}^{(p)*}k^{\nu}_{\lambda} = \delta^p_1(\overset{1}{A}_\lambda A^\nu + \overset{3}{A}_\lambda A^\nu) + \delta^p_2 \overset{3}{A}_\lambda A^\nu \tag{4.21b} $$

$$ {}^{(p)*}k_{\lambda\mu} = 2\delta^p_1 \overset{3}{A}_{(\lambda} \overset{1}{A}_{\mu)} - \delta^p_2 \overset{3}{A}_\lambda \overset{3}{A}_\mu \tag{4.21c} $$

$$ {}^{(p)*}k^{\lambda\nu} = 2\delta^p_1 \underset{1}{A}^{(\lambda} \underset{2}{A}^{\nu)} - \delta^p_2 \underset{2}{A}^\lambda \underset{2}{A}^\nu \tag{4.21d} $$

Theorem 4.18. If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, we have $(r=0, 1, 2, \ldots)$

$$ \overset{(12)r}{T} = 0 \tag{422a} $$

$$ \overset{22r}{T} = 0 \tag{4.22b} $$

$$ \overset{11r}{T}_{\omega\mu\nu} = 2\sum_x T_{12i} \overset{3}{A}_{[\omega} \overset{1}{A}_{\mu]} \overset{x}{A}_\nu \, {}^{(r)*}k^i_x \tag{4.22c} $$

$$ \overset{(20)r}{T}_{\omega\mu\nu} = \sum_x T_{12i} \overset{1}{A}_{[\omega} \overset{3}{A}_{\mu]} \overset{x}{A}_\nu \, {}^{(r)*}k^i_x \tag{4.22d} $$

$$ \overset{(10)r}{T}_{\omega\mu\nu} = \sum_X (T_{12i} \overset{3}{A}_{[\omega} \overset{2}{A}_{\mu]} + T_{23i} \overset{1}{A}_{[\omega} \overset{3}{A}_{\mu]}) \overset{x}{A}_\nu \, {}^{(r)*}k^i_x \tag{4.22e} $$

Theorem 4.19. The following relations hold:

$$ \overset{002}{K} = \overset{(10)1}{K} = \overset{(10)2}{K} = \overset{(12)r}{K} = \overset{22r}{K} = \overset{11r}{K} = \overset{(20)r}{K} = \overset{121}{K} = \overset{022}{K} = 0 \tag{4.23a} $$

$$ \overset{(pq)r}{K} = 0 \qquad \text{if at least one of } p, q, r \text{ is } \geq 3 \qquad (r = 0, 1, 2, \ldots) \tag{4.23b} $$

Theorem 4.20. We have

$$ 2B_{\omega\mu\nu} = K_{\omega\mu\nu} + \overset{200}{K}_{\nu[\omega\mu]} \tag{4.24} $$

Theorem 4.21. The system (2.22) always admits a unique solution, namely

$$ S_{\omega\mu\nu} = \tfrac{1}{2}K_{\omega\mu\nu} + \tfrac{1}{2}\overset{200}{K}_{\nu[\omega\mu]} \tag{4.25} $$

Proof. From (3.12) and (4.23) we have

$$ \overset{110}{B} = \overset{(10)1}{B} = 0 \tag{4.26a} $$

so that

$$ \overset{110}{S} = \overset{(10)1}{S} = 0 \tag{4.26b} $$

Comparing (4.26b) and (2.22), we finally have (4.25). ■

5. 5-*g-UFT. THE FIRST AND SECOND CLASSES

In 5-*g-UFT there are three classes; the first class when $K_4 \neq 0$, the second class when $K_4 = 0$ and $K_2 \neq 0$, and the third class when $K_4 = K_2 = 0$. In this case, relation (2.11b) gives

$$g = 1 + K_2 + K_4$$

In this section we investigate the first two classes of 5-*g-UFT. Hence all considerations in this section are restricted to $n = 5$.

5.1. Basic Vectors and Nonholonomic Frame of Reference

Agreement 5.1. For the simplicity of our discussion, we assume in this and in what follows that

$$K_4 \leq 0 \tag{5.1}$$

Theorem 5.2a. (First class.) In the first class, the basic scalars are given by

$$\begin{aligned} &\underset{1}{M} = -\underset{2}{M} = (-L-K)^{1/2} \neq 0 \\ &\underset{3}{M} = -\underset{4}{M} = (L-K)^{1/2} \neq 0, \qquad \underset{5}{M} = 0 \end{aligned} \tag{5.2}$$

where

$$K = K_2/2, \qquad L = [(K_2/2)^2 - K_4]^{1/2} \tag{5.3}$$

Proof. For the first class of 5-*g-UFT, the characteristics equation (2.15) is reduced to

$$M(M^4 + K_2 M^2 + K_4) = 0 \tag{5.4}$$

from which our assertion follows in virtue of (5.1) and (5.3). ■

Theorem 5.2b. (Second class.) In the second class the basic scalars are given by

$$\underset{1}{M} = -\underset{2}{M} = (-K_2)^{1/2} \neq 0, \qquad \underset{3}{M} = \underset{4}{M} = \underset{5}{M} = 0 \tag{5.5}$$

Proof. Relations (5.5) are simple consequences of substitution of $K_4 = 0$ into (5.2). ■

Theorem 5.3. In the first two classes there are five linearly independent basic vectors $\underset{1}{A}^\nu, \ldots, \underset{5}{A}^\nu$, which have the following properties:

(a) They are defined up to an arbitrary factor of proportionality.
(b) $\underset{1}{A}^\nu, \ldots, \underset{4}{A}^\nu$ are null vectors, while $\underset{5}{A}^\nu$ is nonnull.
(c) $\underset{1}{A}^\nu$ and $\underset{2}{A}^\nu$ are perpendicular to $\underset{3}{A}^\nu$ and $\underset{4}{A}^\nu$. And $\underset{5}{A}^\nu$ is also perpendicular to $\underset{1}{A}^\nu, \ldots, \underset{4}{A}^\nu$.
(d) They satisfy the conditions

$$ {}^*h_{\lambda\mu}\underset{1}{A}^\lambda \underset{2}{A}^\mu \neq 0, \qquad {}^*h_{\lambda\mu}\underset{3}{A}^\lambda \underset{4}{A}^\mu \neq 0 \tag{5.6} $$

Proof. The proof follows by a similar pattern to that of Theorem 4.2. ■

The following theorem immediately follows from the above theorem.

Theorem 5.4. In the first two classes, the factor of proportionality mentioned in Theorem 5.3(a) may be chosen in such a way that the nonholonomic components ${}^*h_{ij}$ and ${}^*h^{ij}$ are given by

$$ (({}^*h_{ij})) = (({}^*h^{ij})) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.7} $$

Relation (3.1) together with (5.7) give the following result:

Theorem 5.5. (First and second classes.) We have

$$ \begin{array}{lllll} \underset{1}{A}^\nu = \overset{2}{A}{}^\nu, & \underset{2}{A}^\nu = \overset{1}{A}{}^\nu, & \underset{3}{A}^\nu = \overset{4}{A}{}^\nu, & \underset{4}{A}^\nu = \overset{3}{A}{}^\nu, & \underset{5}{A}^\nu = \overset{5}{A}{}^\nu \\ \overset{1}{A}_\lambda = \underset{2}{A}_\lambda, & \overset{2}{A}_\lambda = \underset{1}{A}_\lambda, & \overset{3}{A}_\lambda = \underset{4}{A}_\lambda, & \overset{4}{A}_\lambda = \underset{3}{A}_\lambda, & \overset{5}{A}_\lambda = \underset{5}{A}_\lambda \end{array} \tag{5.8} $$

As an application of the nonholonomic frame of reference constructed in the above, we have the following theorem. The representations are consequences of (2.17b), (5.2), (5.7), and (5.8).

Theorem 5.6. In the first two classes, the tensors $*h_{\lambda\mu}$, ${}^{(p)*}k_\lambda^\nu$, ${}^{(p)*}k_{\lambda\mu}$, and ${}^{(p)*}k^{\lambda\nu}$ may be expressed in terms of basic vectors as follows:

$$*h_{\lambda\mu} = 2\overset{1}{A}_{(\lambda}\overset{2}{A}_{\mu)} + 2\overset{3}{A}_{(\lambda}\overset{4}{A}_{\mu)} + \overset{5}{A}_\lambda\overset{5}{A}_\mu \tag{5.9a}$$

$${}^{(p)*}k_\lambda^\nu = \underset{1}{M}^p[\overset{1}{A}_\lambda\underset{1}{A}^\nu + (-1)^p\overset{2}{A}_\lambda\underset{2}{A}^\nu] + \underset{3}{M}^p[\overset{3}{A}_\lambda\underset{3}{A}^\nu + (-1)^p\overset{4}{A}_\lambda\underset{4}{A}^\nu] \tag{5.9b}$$

$${}^{(p)*}k_{\lambda\mu} = \begin{cases} 2\underset{1}{M}^p\overset{1}{A}_{(\lambda}\overset{2}{A}_{\mu)} + 2\underset{3}{M}^p\overset{3}{A}_{(\lambda}\overset{4}{A}_{\mu)} & \text{if } p \text{ is even} \\ 2\underset{1}{M}^p\overset{1}{A}_{[\lambda}\overset{2}{A}_{\mu]} + 2\underset{3}{M}^p\overset{3}{A}_{[\lambda}\overset{4}{A}_{\mu]} & \text{if } p \text{ is odd} \end{cases} \tag{5.9c}$$

$${}^{(p)*}k^{\lambda\nu} = \begin{cases} 2\underset{1}{M}^p\underset{1}{A}^{(\lambda}\underset{2}{A}^{\nu)} + 2\underset{3}{M}^p\underset{3}{A}^{(\lambda}\underset{4}{A}^{\nu)} & \text{if } p \text{ is even} \\ 2\underset{1}{M}^p\underset{1}{A}^{[\lambda}\underset{2}{A}^{\nu]} + 2\underset{3}{M}^p\underset{3}{A}^{[\lambda}\underset{4}{A}^{\nu]} & \text{if } p \text{ is odd} \end{cases} \tag{5.9d}$$

5.2. Recurrence Relations

In this section we derive several useful and powerful recurrence relations in addition to (3.3) and (3.6), which now in 5-$*g$-UFT take the form ($p = 0, 1, 2, \ldots$)

$${}^{(p+5)*}k_\lambda^\nu = -K_2\,{}^{(p+3)*}k_\lambda^\nu - K_4\,{}^{(p+1)*}k_\lambda^\nu \quad \text{for first class} \tag{5.10a}$$

$${}^{(p+3)*}k_\lambda^\nu = -K_2\,{}^{(p+1)*}k_\lambda^\nu \quad \text{for second class.} \tag{5.10b}$$

The following two theorems are simple consequences of (5.2) and (5.5).

Theorem 5.7a. (First class.). In the first class the basic scalars $\underset{x}{M}$ satisfy

$$\underset{1}{M} + \underset{2}{M} = \underset{3}{M} + \underset{4}{M} = 0, \tag{5.11a}$$

$$\underset{1}{M}\underset{5}{M} = \underset{2}{M}\underset{5}{M} = \underset{3}{M}\underset{5}{M} = \underset{4}{M}\underset{5}{M} = 0 \tag{5.11b}$$

$$\underset{1}{M}^2\underset{3}{M}^2 = \underset{1}{M}^2\underset{4}{M}^2 = \underset{2}{M}^2\underset{3}{M}^2 = \underset{2}{M}^2\underset{4}{M}^2 = K_4 \tag{5.11c}$$

$$\underset{1}{M}^2 + \underset{3}{M}^2 = \underset{1}{M}^2 + \underset{4}{M}^2 = \underset{2}{M}^2 + \underset{3}{M}^2 = \underset{2}{M}^2 + \underset{4}{M}^2 = -K_2 \tag{5.11d}$$

Theorem 5.7b. (Second class.) In the second class the basic scalars $\underset{x}{M}$ satisfy $(x, y=3, 4, 5)$

$$\underset{1}{M}+\underset{2}{M}=\underset{x}{M}+\underset{y}{M}=0 \tag{5.12a}$$

$$\underset{1}{M}\underset{2}{M}=K_2, \qquad \underset{1}{M}\underset{x}{M}=\underset{2}{M}\underset{x}{M}=\underset{x}{M}\underset{y}{M}=0 \tag{5.12b}$$

In virtue of Theorems (5.7a) and (5.7b) we have

Theorem 5.8a. (First class.) In the first class, the following identities hold for all values of x and y when $x \neq y$:

$$\underset{x}{M}^{(4}\underset{y}{M}^{1)}=-\underset{x}{M}^{(3}\underset{y}{M}^{2)}-K_2\,\underset{x}{M}^{(2}\underset{y}{M}^{1)} \tag{5.13a}$$

$$\underset{x}{M}^{(4}\underset{y}{M}^{3)}=K_4\,\underset{x}{M}^{(2}\underset{y}{M}^{1)} \tag{5.13b}$$

$$\underset{x}{M}^{4}\underset{y}{M}=(K_4)^2\underset{x}{M}^{2}\underset{y}{M}^{2}+K_2\,\underset{x}{M}^{3}\underset{y}{M}^{3}+2K_4\,\underset{x}{M}^{(3}\underset{y}{M}^{1)} \tag{5.13c}$$

$$2\underset{x}{M}^{(4}\underset{y}{M}^{2)}=-\underset{x}{M}^{3}\underset{y}{M}^{3}-K_2\,\underset{x}{M}^{2}\underset{y}{M}^{2}+K_4\,\underset{x}{M}\underset{y}{M} \tag{5.13d}$$

Furthermore, we also have

$$K_4\,\underset{x}{M}^{(4}\underset{y}{M}^{1)}+K_2\,\underset{x}{M}^{(4}\underset{y}{M}^{3)}=-K_4\,\underset{x}{M}^{(2}\underset{y}{M}^{3)} \tag{5.14a}$$

$$K_2\,\underset{x}{M}^{4}\underset{y}{M}^{4}+2K_4\,\underset{x}{M}^{(4}\underset{y}{M}^{2)}=[(K_2)^2-K_4]\underset{x}{M}^{3}\underset{y}{M}^{3}+2K_2K_4\,\underset{x}{M}^{(3}\underset{y}{M}^{1)}$$
$$+(K_4)^2\,\underset{x}{M}\underset{y}{M} \tag{5.14b}$$

$$\underset{x}{M}^{4}\underset{y}{M}^{4}+2K_2\,\underset{x}{M}^{(4}\underset{y}{M}^{2)}=[K_4-(K_2)^2]\underset{x}{M}^{2}\underset{y}{M}^{2}+2K_4\underset{x}{M}^{(3}\underset{y}{M}^{1)}$$
$$+K_2K_4\,\underset{x}{M}\underset{y}{M} \tag{5.14c}$$

Proof. In order to prove (5.13a), consider

$$B\overset{\text{def}}{=}\underset{x}{M}\underset{y}{M}(\underset{x}{M}+\underset{y}{M})(\underset{x}{M}^2+\underset{y}{M}^2)$$

In virtue of (5.11d), we have

$$B=-2K_2\,\underset{x}{M}^{(2}\underset{y}{M}^{1)} \tag{5.15a}$$

On the other hand, direct algebraic calculation gives

$$B = 2M_x^{(4}M_y^{1)} + 2M_x^{(3}M_y^{2)} \tag{5.15b}$$

Thus, comparison of (5.15a) with (5.15b) yields (5.13a). For the proof of (5.13b), we consider

$$M_x M_y (M_x + M_y) M_x^2 M_y^2$$

and use the same pattern as the proof of (5.13a), making use of (5.11c). In order to prove (5.13c), put

$$C \overset{\text{def}}{=} M_x M_y (M_x + M_y)(M_x^4 M_y + M_x M_y^4)$$

Making use of (5.13a) and (5.10a), we have

$$\begin{aligned} C &= -4M_x^{(2}M_y^{1)}(M_x^{(3}M_y^{2)} + K_2\, M_x^{(2}M_y^{1)}) \\ &= -2(M_x^{(5}M_y^{3)} + M_x^4 M_y^4 + K_2\, M_x^{(4}M_y^{2)} + K_2\, M_x^3 M_y^3) \\ &= 2(K_2\, M_x^{(3} + K_4\, M_x^{(1})M_y^{3)} - 2M_x^4 M_y^4 - 2K_2\, M_x^{(4}M_y^{2)} - 2K_2\, M_x^3 M_y^3 \\ &= 2K_4\, M_x^{(1}M_y^{3)} - 2M_x^4 M_y^4 - 2K_2\, M_x^{(4}M_y^{2)} \end{aligned} \tag{5.16a}$$

On the other hand, direct algebraic calculation gives

$$\begin{aligned} C &= 2M_x^{(6}M_y^{2)} + 2M_x^{(5}M_y^{3)} \\ &= -2(K_2\, M_x^{(4} + K_4\, M_x^{(2})M_y^{2)} - 2(K_2\, M_x^{(3} + K_4\, M_x^{(1})M_y^{3)} \\ &= -2K_2\, M_x^{(4}M_y^{2)} - 2K_4\, M_x^2 M_y^2 - 2K_2\, M_x^3 M_y^3 - 2K_4\, M_x^{(1}M_y^{3)} \end{aligned} \tag{5.16b}$$

Comparing (5.16a) and (5.16b), one gets (5.13c). Finally, consider the following form in order to prove (5.13d):

$$D \overset{\text{def}}{=} M_x M_y (M_x + M_y)(M_x^4 M_y^3 + M_x^3 M_y^4)$$

The relation (5.13b) gives

$$D = K_4(M_x^2 M_y + M_x M_y^2) = 2K_4\, M_x^{(4}M_y^{2)} + 2K_4\, M_x^3 M_y^3 \tag{5.17a}$$

while by a direct algebraic calculation making use of (5.10a) and (5.13c) we have

$$\begin{aligned} D &= 2\underset{x}{M}^{(6}\underset{y}{M}^{4)} + 2\underset{x}{M}^{5}\underset{y}{M}^{5} \\ &= -2(K_2\underset{x}{M}^{(4} + K_4\underset{x}{M}^{(2)})\underset{y}{M}^{4)} + 2(K_2\underset{x}{M}^{3} + K_4\underset{x}{M})(K_2\underset{y}{M}^{3} + K_4\underset{y}{M}) \\ &= -2K_2(K_2\underset{x}{M}^{3}\underset{y}{M}^{3} + K_4\underset{x}{M}^{2}\underset{y}{M}^{2} + 2K_4\underset{x}{M}^{(3}\underset{y}{M}^{1)}) - 2K_2K_4\underset{x}{M}^{(2}\underset{y}{M}^{4)} \\ &\quad + 2(K_2)^2\underset{x}{M}^{3}\underset{y}{M}^{3} + 2(K_4)^2\underset{x}{M}\underset{y}{M} + 4K_2K_4\underset{x}{M}^{(3}\underset{y}{M}^{1)} \\ &= -2K_2K_4\underset{x}{M}^{2}\underset{y}{M}^{2} - 2K_4\underset{x}{M}^{(2}\underset{y}{M}^{4)} + 2(K_4)^2\underset{x}{M}\underset{y}{M} \end{aligned} \tag{5.17b}$$

Since $K_4 \neq 0$, comparison of (5.17a) and (5.17b) gives (5.13d). The relations (5.14) are direct consequences of (5.13). ■

Theorem 5.8b. (Second class.) In the second class the following identities hold for all values of x and y when $x \neq y$:

$$\underset{x}{M}^{2}\underset{y}{M}^{2} = K_2\underset{x}{M}\underset{y}{M} \tag{5.18a}$$

$$\underset{x}{M}^{(2}\underset{y}{M}^{1)} = 0 \tag{5.18b}$$

Proof. The assertions in this theorem are direct consequences of (5.12). ■

Now we are in a position to establish the following recurrence relations, which may be proved simultaneously.

Theorem 5.9a. (First class.) If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then the following recurrence relations hold in the first class:

$$\overset{(41)r}{T} = -\overset{(32)r}{T} - K_2\overset{(21)r}{T} \tag{5.19a}$$

$$\overset{(43)r}{T} = K_4\overset{(21)r}{T} \tag{5.19b}$$

$$\overset{44r}{T} = K_4\overset{22r}{T} + K_2\overset{33r}{T} + 2K_4\overset{(31)r}{T} \tag{5.19c}$$

$$2\overset{(42)r}{T} = -\overset{33r}{T} - K_2\overset{22r}{T} + K_4\overset{11r}{T} \tag{5.19d}$$

Furthermore, the following identities also hold in the first class:

$$K_4\overset{(41)r}{T} + K_2\overset{(43)r}{T} = -K_4\overset{(23)r}{T} \tag{5.20a}$$

$$K_2\overset{44r}{T} + 2K_4\overset{(42)r}{T} = [(K_2)^2 - K_4]\overset{33r}{T} + 2K_2K_4\overset{(31)r}{T} + (K_4)^2\overset{11r}{T} \tag{5.20b}$$

$$\overset{44r}{T}+2K_2\overset{(42)r}{T}=[K_4-(K_2)^2]\overset{22r}{T}+2K_4\overset{(31)r}{T}+K_2K_4\overset{11r}{T} \tag{5.20c}$$

Theorem 5.9b. (Second class.) If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then the following recurrence relations hold in the second class:

$$\overset{(21)r}{T}=0 \tag{5.21a}$$

$$\overset{(22)r}{T}=K_2\overset{11r}{T} \tag{5.21b}$$

Proof. The proofs of the above two theorems follow from (3.2a), (5.13), and (5.18). For example, the relation (5.19a) may be obtained as follows:

$$\begin{aligned}\overset{(41)r}{T}=\overset{(41)r}{T}_{\omega\mu\nu}&=\sum_{x,y,z}T_{xyz}\underset{x}{M}{}^{(4}\underset{y}{M}{}^{1)}\underset{z}{M}{}^{r}\overset{x}{A}_{\omega}\overset{y}{A}_{\mu}\overset{z}{A}_{\nu}\\&=\sum_{x,y,z}T_{xyz}(-\underset{x}{M}{}^{(3}\underset{y}{M}{}^{2)}-K_2\,\underset{x}{M}{}^{(2}\underset{y}{M}{}^{1)})\underset{z}{M}{}^{r}\overset{x}{A}_{\omega}\overset{y}{A}_{\mu}\overset{z}{A}_{\nu}\\&=-\overset{(32)r}{T}_{\omega\mu\nu}-K_2\overset{(21)r}{T}_{\omega\mu\nu}=-\overset{(32)r}{T}-K_2\overset{(21)r}{T}\end{aligned}$$

5.3. Einstein Connection $\Gamma^{\nu}_{\lambda\mu}$

In this section we obtain surveyable representations of $\Gamma^{\nu}_{\lambda\mu}$ in terms of $^*g^{\lambda\nu}$, using the recurrence relations derived in the preceeding section.

Theorem 5.10a. (First class.) A necessary and sufficient condition for the existence and uniqueness of the solution of (2.8a) in the first class is

$$gAB(C^2-4K_4D^2)\neq 0 \tag{5.22}$$

where

$$A=1-K_2+K_4,\qquad B=1-K_4 \tag{5.23}$$

$$C=1-K_2+5K_4,\qquad D=K_2-2 \tag{5.24}$$

Proof. In virtue of (5.2) and (5.3), the symmetric scalars $\underset{xyz}{M}$ defined by (2.26) take the values shown in Table I. It may be easily verified that the product of two factors in the first row in Table I is g, that of the five factors in the second row is $(1-K_2+K_4)(1-K_4)$, and that of the four factors in the third row is $(1-K_2+5K_4)^2-4K_4(K_2-2)^2$. Hence we have proved our assertion (5.22) in virtue of (2.26), (5.23), and (5.24).

Theorem 5.10b. (Second class.) A necessary and sufficient condition for the existence and uniqueness of the solution of (2.8a) in the second call is

$$1-(K_2)^2\neq 0 \tag{5.25}$$

Table I

Values of indices x, y, z	$\underset{xyz}{M}$
Two of the indices x, y, z are 1, 2 or 3, 4	$1+K+L,\ 1+K-L$
At least one of x, y, z is 5 and no two take the values 1, 2 or 3, 4	$1-K+L,\ 1-K-L,\ 1+\sqrt{K_4},\ 1-\sqrt{K_4},\ 1$
The remaining cases	$1-K-L-2\sqrt{K_4},\ 1-K+L-2\sqrt{K_4},$ $1-K-L+2\sqrt{K_4},\ 1-K+L+2\sqrt{K_4}$

Proof. The assertion of this theorem may be obtained by simply substituting $K_4=0$ into (5.22) and (5.23). ■

Theorem 5.11a. (First class.) The system of equations (2.22) is reduced to the following 25 equations:

$$B = S + \overset{110}{S} + 2\overset{(10)1}{S}$$

$$\overset{(10)1}{B} = \overset{(10)1}{S} + \overset{(21)1}{S} + \overset{(20)2}{S} + \overset{112}{S}$$

$$\overset{(12)1}{B} = \overset{(12)1}{S} + \overset{(23)1}{S} + \overset{222}{S} + \overset{(13)2}{S}$$

$$\overset{(20)2}{B} = \overset{(20)2}{S} + \overset{(31)2}{S} + \overset{(30)3}{S} + \overset{(21)3}{S}$$

$$2\overset{(23)1}{B} = 2\overset{(23)1}{S} + 2K_4\overset{(21)1}{S} + \overset{332}{S} + K_4\overset{112}{S} - K_2\overset{222}{S}$$

$$2\overset{(13)2}{B} = 2\overset{(13)2}{S} + K_4\overset{112}{S} - K_2\overset{222}{S} - \overset{332}{S} - 2K_2\overset{(21)3}{S}$$

$$\overset{(30)3}{B} = \overset{(30)3}{S} - K_2\overset{(21)3}{S} - \overset{(32)3}{S} + \overset{(40)4}{S} + \overset{(31)4}{S}$$

$$\overset{(21)3}{B} = \overset{(21)3}{S} + \overset{(32)3}{S} + \overset{(31)4}{S} + \overset{224}{S}$$

$$2\overset{(32)3}{B} = 2\overset{(32)3}{S} + 2K_4\overset{(21)3}{S} + K_4\overset{114}{S} - K_2\overset{224}{S} + \overset{334}{S}$$

$$\overset{(40)4}{B} = \overset{(40)4}{S} - K_2\overset{(31)4}{S} - K_4\overset{114}{S} + K_2\overset{(30)3}{S} + K_2K_4\overset{(10)3}{S} + K_2K_4\overset{(30)1}{S}$$
$$+ K_4\overset{(10)1}{S} + K_2\overset{(21)3}{S} + K_2K_4\overset{(21)1}{S} + K_2\overset{(32)3}{S} + K_4\overset{(32)1}{S}$$

$$2\overset{(31)4}{B} = 2\overset{(31)4}{S} + 2K_2\overset{(21)3}{S} + 2K_2K_4\overset{(21)1}{S} + K_4\overset{114}{S} - K_2\overset{224}{S} - \overset{334}{S}$$

$$\overset{(10)3}{B} = \overset{(10)3}{S} + \overset{(21)3}{S} + \overset{(20)4}{S} + \overset{114}{S}$$

$$\overset{(30)1}{B} = \overset{(30)1}{S} - K_2 \overset{(21)1}{S} - \overset{(32)1}{S} + \overset{(40)2}{S} + \overset{(31)2}{S}$$

$$\overset{(20)4}{B} = \overset{(20)4}{S} + \overset{(31)4}{S} - K_2 \overset{(30)3}{S} - K_4 \overset{(30)1}{S} - K_2 \overset{(21)3}{S} - K_4 \overset{(21)1}{S}$$

$$\overset{(40)2}{B} = \overset{(40)2}{S} - K_2 \overset{(31)2}{S} - K_4 \overset{112}{S} - K_2 \overset{(30)3}{S} - K_4 \overset{(10)3}{S} - K_2 \overset{(21)3}{S} - \overset{(32)3}{S} \tag{5.26}$$

$$\overset{110}{B} = \overset{110}{S} + \overset{220}{S} + 2 \overset{(21)1}{S}$$

$$\overset{112}{B} = \overset{112}{S} + \overset{222}{S} + 2 \overset{(21)3}{S}$$

$$\overset{222}{B} = \overset{222}{S} + \overset{332}{S} + 2 \overset{(32)3}{S}$$

$$\overset{332}{B} = (1+K_2) \overset{332}{S} + K_4 \overset{222}{S} + 2K_4 \overset{(21)3}{S} + 2K_4 \overset{(31)2}{S}$$

$$\overset{224}{B} = \overset{224}{S} + \overset{334}{S} - 2K_2 \overset{(32)3}{S} - 2K_4 \overset{(32)1}{S}$$

$$\overset{114}{B} = \overset{114}{S} + \overset{224}{S} - 2K_2 \overset{(21)3}{S} - 2K_4 \overset{(21)1}{S}$$

$$\overset{334}{B} = (1+K_2) \overset{334}{S} + K_4 \overset{224}{S} + 2K_4 \overset{(31)4}{S} - 2K_2K_4 \overset{(21)3}{S} - 2(K_4)^2 \overset{(21)1}{S}$$

$$\overset{220}{B} = \overset{220}{S} + \overset{330}{S} + 2 \overset{(32)1}{S}$$

$$\overset{330}{B} = (1+K_2) \overset{330}{S} + K_4 \overset{220}{S} + 2K_4 \overset{(31)0}{S} + 2K_4 \overset{(21)1}{S}$$

$$2 \overset{(31)0}{B} = 2 \overset{(31)0}{S} + K_4 \overset{110}{S} - K_2 \overset{220}{S} - \overset{330}{S} - 2K_2 \overset{(21)1}{S}$$

Proof. This assertion follows from (3.11) using (5.10a), (5.19), and (5.20). ■

Theorem 5.11b. (Second class.) The system of equations (2.22) is reduced to the following five equations:

$$B = S + 2 \overset{(10)1}{S} + \overset{110}{S}$$

$$\overset{(10)1}{B} = \overset{(10)1}{S} + \overset{(20)2}{S} + \overset{112}{S}$$

$$\overset{(20)2}{B} = (K_2)^2 \overset{(10)1}{S} + \overset{(20)2}{S} - K_2 \overset{112}{S} \tag{5.27}$$

$$\overset{110}{B} = (1+K_2) \overset{110}{S}$$

$$\overset{112}{B} = (1+K_2) \overset{112}{S}$$

Proof. The above equations may be obtained from (3.11), using (5.10b) and (5.21).

Theorem 5.12a. (First class.) If the condition (5.22) is satisfied, the unique solution of (2.22) is given by

$$\begin{aligned} gAB(C^2-4K_4D^2)(S-B) = {} & (1-K_2+5K_4)\underset{1}{B}+2[1-2K_2+(K_2)^2-5K_4]\underset{2}{B} \\ & +2(K_2-2)\underset{3}{B} \end{aligned} \tag{5.28a}$$

where

$$\begin{aligned} \underset{1}{B} = {} & -\overset{222}{B}+2(\overset{(20)2}{B}+K_4\overset{(20)4}{B}-\overset{(10)1}{B}+\overset{112}{B})+(K_2-1) \\ & [\overset{220}{B}+K_4\overset{114}{B}+3K_2(K_4)^2\overset{224}{B}] \\ & +K_4[2\overset{220}{B}-K_4B-\overset{330}{B}+2(K_2)^2\overset{(31)4}{B}+K_2K_4\overset{(40)2}{B}+4\overset{(10)3}{B} \\ & +K_2(\overset{112}{B}+2\overset{(10)3}{B}-4K_4\overset{(10)1}{B}) \\ & +[(K_2)^2-1+K_4](2\overset{(10)1}{B}-\overset{110}{B}+K_4\overset{(20)4}{B}) \\ & +(K_2-1+2K_4)[\overset{220}{B}+K_4\overset{332}{B}+2(K_2)^2K_4\overset{330}{B}] \end{aligned} \tag{5.28b}$$

$$\begin{aligned} \underset{2}{B} = {} & 2[(K_4)^2B+\overset{(12)3}{B}+K_4\overset{(10)3}{B}+(K_4)^2\overset{(30)1}{B}-K_4\overset{(20)2}{B}]+K_2\overset{112}{B} \\ & +(1+K_2)[2K_4\overset{(10)1}{B}+(K_4)^2\overset{334}{B}+3K_4\overset{(40)4}{B}]-\overset{222}{B} \\ & -K_4[\overset{220}{B}+K_2\overset{(31)0}{B}+2(K_2)^2\overset{(13)2}{B}+K_2K_4\overset{(30)3}{B}] \\ & +(1+K_2)(1+3K_4)(2\overset{(12)1}{B}-\overset{110}{B}) \end{aligned} \tag{5.28c}$$

$$\begin{aligned} \underset{3}{B} = {} & (K_4)^2B+2\overset{(12)3}{B}+[2K_4-(K_2)^2]\overset{112}{B}+K_2[\overset{220}{B}+(K_4)^2\overset{(31)0}{B}] \\ & +K_4[K_2\overset{220}{B}-2\overset{(12)1}{B}+2\overset{(20)2}{B}-(K_2)^2\overset{(30)3}{B}+K_2K_4\overset{(32)3}{B}] \\ & +K_4[2+2K_2-(K_2)^2](\overset{110}{B}+K_2\overset{330}{B}-K_4\overset{334}{B}) \\ & -K_4(1+K_2)[2\overset{(10)3}{B}-\overset{220}{B}-K_2\overset{(10)1}{B}+(K_2)^2\overset{(23)1}{B}]. \end{aligned} \tag{5.28d}$$

Proof. Equations (5.28a)-(5.28d) represent the solution of the system (5.26). This solution was obtained by using the Gauss-Jordan elimination method, employing the MV/8000 II Super-Mini-Computer at Jeonju University. ■

Theorem 5.12b. (Second class.) If the condition (5.25) is satisfied, the unique solution of (2.22) is given by

$$[1-(K_2)^2](S-B) = -2 \overset{(10)1}{B} + (K_2-1) \overset{110}{B} + 2 \overset{(20)2}{B} + 2 \overset{112}{B} \tag{5.29}$$

Proof. Equation (5.29) is the solution of the system (5.27). ■

Now that we have obtained the tensors $S^{\nu}_{\lambda\mu}$ in terms of $*g^{\lambda\nu}$, it is possible to determine the tensor $U^{\nu}_{\lambda\mu}$ by (2.21) and the connection $\Gamma^{\nu}_{\lambda\mu}$ by only substituting for S and U into (2.20).

6. THE THIRD CLASS OF n-$*g$-UFT FOR $n \geq 4$

In this section we investigate the third class of n-$*g$-UFT. All considerations in this section are for general $n \geq 4$.

6.1. Basic Vectors and Nonholonomic Frame of Reference

Theorem 6.1. The basic scalars $\underset{x}{M}$ are given by

$$\underset{1}{M} = \cdots = \underset{n}{M} = 0 \tag{6.1}$$

Proof. For the third class of n-$*g$-UFT, the characteristic equation (2.15) is reduced to

$$M^n = 0 \tag{6.2}$$

from which our assertion follows. ■

The following theorem immediately follows from (2.19b).

Theorem 6.2. The nonholonomic components $*h_{ij}$ and $*h^{ij}$ are given by the matrix equation

$$((*h_{ij})) = ((*h^{ij})) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & & & \\ 0 & -1 & 0 & 0 & 0 & & & \\ 0 & 0 & -1 & 0 & 0 & & 0 & \\ 1 & 0 & 0 & 0 & 0 & & & \\ 0 & 0 & 0 & 0 & -1 & & & \\ & & & & & -1 & & \\ & & 0 & & & & \ddots & \\ & & & & & & & -1 \end{pmatrix} \tag{6.3}$$

The relation (3.1) together with (6.3) give the following result:

Theorem 6.3. We have ($i = 2, 3, 5, \ldots, n$)

$$\underset{1}{A}^{\nu} = \overset{4}{A}{}^{\nu}, \qquad \underset{4}{A}^{\nu} = \overset{1}{A}{}^{\nu}, \qquad \underset{i}{A}^{\nu} = -\overset{i}{A}{}^{\nu}, \qquad \overset{1}{A}_{\lambda} = \underset{4}{A}_{\lambda},$$

$$\overset{4}{A}_{\lambda} = \underset{1}{A}_{\lambda}, \qquad \overset{i}{A}_{\lambda} = -\underset{i}{A}_{\lambda} \tag{6.4}$$

As an application of the nonholonomic frame of reference constructed in the above, we have the following:

Theorem 6.4. The tensors $*h_{\lambda\mu}$, ${}^{(p)*}k_{\lambda}^{\nu}$, ${}^{(p)*}k_{\lambda\mu}$, and ${}^{(p)*}k^{\lambda\nu}$ may be expressed in terms of basic vectors as follows ($p = 1, 2, \ldots$):

$$*h_{\lambda\mu} = 2\overset{1}{A}_{(\lambda}\overset{4}{A}_{\mu)} - \sum_{x=2,3,5}^{n} \overset{x}{A}_{\lambda}\overset{x}{A}_{\mu} \tag{6.5a}$$

$${}^{(p)*}k_{\lambda}^{\nu} = \delta_1^p(\overset{1}{A}_{\lambda}\underset{2}{A}^{\nu} + \overset{2}{A}_{\lambda}\underset{4}{A}^{\nu}) + \delta_2^p\overset{1}{A}_{\lambda}\underset{4}{A}^{\nu} \tag{6.5b}$$

$${}^{(p)*}k_{\lambda\mu} = 2\delta_1^p\overset{2}{A}_{[\lambda}\overset{1}{A}_{\mu]} + \delta_2^p\overset{1}{A}_{\lambda}\overset{1}{A}_{\mu} \tag{6.5c}$$

$${}^{(p)*}k^{\lambda\mu} = 2\delta_1^p\underset{4}{A}^{[\lambda}\underset{2}{A}^{\nu]} + \delta_2^p\underset{4}{A}^{\lambda}\underset{4}{A}^{\nu} \tag{6.5d}$$

Proof. The relations (6.5a)-(6.5d) follow from (2.17b) in virtue of (6.3), (6.4), and (2.19c). ■

6.2. Recurrence Relations and the Tensor $\overset{(pq)r}{K}_{\omega\mu\nu}$

In this section we derive several recurrence relations in addition to (3.10) and investigate the properties of the tensor $\overset{(pq)r}{T}_{\omega\mu\nu}$, which we need in the next section.

Theorem 6.5. (Recurrence relations.) If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then the following recurrence relations hold in the third class of n-$*g$-UFT:

$$\overset{(12)r}{T} = 0 \tag{6.6a}$$

$$\overset{22r}{T} = 0 \tag{6.6b}$$

Proof. The proof of (6.6a) and (6.6b) is based on the following relations:

$$\begin{aligned}\overset{(pq)r}{T}_{\omega\mu\nu} &= \sum_{x,y,z} \overset{(pq)r}{T}_{xyx}\ \overset{x}{A}_{\omega}\overset{y}{A}_{\mu}\overset{z}{A}_{\nu} \\ &= \tfrac{1}{2}\sum_{x,y,z}^{9} T_{ijk}({}^{(p)*}k_x^i\,{}^{(q)*}k_y^j + {}^{(q)*}k_x^i\,{}^{(p)*}k_y^j)\,{}^{(r)*}k_z^k\overset{x}{A}_{\omega}\overset{y}{A}_{\mu}\overset{z}{A}_{\nu}\end{aligned} \tag{6.7}$$

In virtue of (6.7), (2.19c), and the skew-symmetry of the tensor $T_{\omega\mu\nu}$, we can derive (6.6a) and (6.6b) in the following way:

$$\overset{(12)r}{T}_{\omega\mu\nu}=\tfrac{1}{2}\sum_{x,y,z} T_{ijk}({}^{*}k_x^i\,{}^{(2)*}k_y^j+{}^{(2)*}k_x^i\,{}^{*}k_y^j)\,{}^{(r)*}k_z^k\overset{x}{A}_\omega\overset{y}{A}_\mu\overset{z}{A}_\nu$$

$$=\tfrac{1}{2}\sum_z (T_{24k}+T_{42k})\,{}^{(r)*}k_z^k\overset{1}{A}_\omega\overset{1}{A}_\mu\overset{z}{A}_\nu=0$$

$$\overset{22r}{T}_{\omega\mu\nu}=\sum_{x,y,z} T_{ijk}\,{}^{(2)*}k_x^i\,{}^{(2)*}k_y^j\,{}^{(r)*}k_z^k\,\overset{x}{A}_\omega\overset{y}{A}_\mu\overset{z}{A}_\nu$$

$$=\sum_z T_{44k}\,{}^{(r)*}k_z^k\,\overset{x}{A}_\omega\overset{y}{A}_\mu\overset{z}{A}_\nu=0$$

Theorem 6.6. The following relations hold ($r=0,1,2,\dots$)

$$K_{i[24]}=K_{434}=K_{343}=0 \tag{6.8a}$$

$$\overset{111}{K}=\overset{121}{K}=\overset{(02)2}{K}=\overset{112}{K}=\overset{122}{K}=\overset{22r}{K}=\overset{(21)r}{K}=0 \tag{6.8b}$$

$$\overset{(pq)r}{K}=0 \qquad \text{if at least one of } p,\ q,\ r \text{ is } \geq 3 \tag{6.8c}$$

Proof. Relation (6.8a) is a direct consequence of (2.10e) and (2.19c). The relations (6.8b) are consequences of (6.7) and (2.19c). For example, the second relation of (6.8b) may be proved as follows:

$$\overset{121}{K}_{\omega\mu\omega}=\sum_{x,y,z} K_{ijk}\,{}^{*}k_x^i\,{}^{(2)*}k_y^j\,{}^{*}k_z^k\overset{x}{A}_\omega\overset{y}{A}_\mu\overset{z}{A}_\nu$$

$$=K_{242}\overset{1}{A}_\omega\overset{1}{A}_\mu\overset{1}{A}_\nu+K_{244}\overset{1}{A}_\omega\overset{1}{A}_\mu\overset{2}{A}_\nu+K_{442}\overset{2}{A}_\omega\overset{1}{A}_\mu\overset{1}{A}_\nu+K_{444}\overset{2}{A}_\omega\overset{1}{A}_\mu\overset{2}{A}_\nu$$

$$=K_{224}\overset{1}{A}_\omega\overset{1}{A}_\mu\overset{1}{A}_\nu-K_{442}\overset{1}{A}_\omega\overset{1}{A}_\mu\overset{2}{A}_\nu=0$$

The relation (6.8c) follows easily from (6.7), using (3.10). ■

6.3. Einstein Connection $\Gamma^\nu_{\lambda\mu}$

In this section we obtain a simple tensorial representation of $\Gamma^\nu_{\lambda\mu}$ in terms of ${}^{*}g^{\lambda\nu}$, using the recurrence relations and results derived in the preceding section.

Theorem 6.7. We have

$$\overset{112}{B}=\overset{(20)2}{B}=0 \tag{6.9a}$$

$$\overset{112}{S}=\overset{(20)2}{S}=0 \tag{6.9b}$$

Proof. Relation (6.9a) is obtained from (3.12) by using (6.8b) and (6.8c). Relation (6.9b) also may be obtained from (3.11) by using (3.10) and (6.9a). ■

Theorem 6.8. The solution of (2.22) is given by

$$S = B - 3 \overset{(110)}{B} \tag{6.10}$$

Proof. From (6.6a), (3.11), and (6.9b), we have

$$\overset{(110)}{S} = \overset{(110)}{B} \tag{6.11}$$

Comparing (2.22) and (6.11), one gets (6.10). ■

Theorem 6.9. The tensor $U^{\nu}_{\lambda\mu}$ is given by a linear combination of $\overset{(pq)r}{B}_{\lambda\mu\nu}$:

$$U_{\nu\lambda\mu} = \overset{[10]0}{B}_{\lambda\mu\nu} + \overset{[02]1}{B}_{\lambda\mu\nu} + \overset{[12]0}{B}_{\lambda\mu\nu} + 2(\overset{(01)0}{B}_{\nu(\lambda\mu)} - \overset{[02]1}{B}_{\nu(\lambda\mu)}) \tag{6.12}$$

Proof. Relation (6.12) follows from (2.21), making use of (6.8b), (6.8c), and (6.10). ■

Now that we have obtained the tensors $S^{\nu}_{\lambda\mu}$ and $U^{\nu}_{\lambda\mu}$ in terms of $*g^{\lambda\mu}$, it is possible to determine the connection $\Gamma^{\nu}_{\lambda\mu}$ by substituting for S and U into (2.20).

Theorem 6.10. The Einstein connection $\Gamma^{\nu}_{\lambda\mu}$ in n-*g-UFT is given by

$$\Gamma^{\nu}_{\lambda\mu} = *\{{}^{\nu}_{\lambda\mu}\} + B^{\nu}_{\lambda\mu} - 3\overset{(110)}{B}{}^{\nu}_{\lambda\mu} + \overset{[10]0}{B}{}^{\nu}_{\lambda\mu} + \overset{[02]1}{B}{}^{\nu}_{\lambda\mu} + \overset{[12]0}{B}{}^{\nu}_{\lambda\mu} + 2(\overset{(01)0}{B}{}^{\nu}_{(\lambda\mu)} - \overset{[02]1}{B}{}^{\nu}_{(\lambda\mu)}) \tag{6.13}$$

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